



## Lecture 7 (optics) field quantization and quantum coherence

- field quantization  
- quantum coherence

$\vec{A}(x, y, z, t)$   
electromagnetic  
( $U_1, U_2, \dots$ )

Maxwell's Equation

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})\end{aligned}$$

electric field  
 $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$   
magnetic field  
 $\vec{B} = \vec{\nabla} \times \vec{A}$

quantization

mode

②  
matter  
quantization

1. dynamical variables  $\rightarrow$  operators  
 $x, p \rightarrow \hat{x}, \hat{p}$

2. ballistic motion  $\rightarrow$  probability wave (function)  
(state)

$\hat{x} \rightarrow \psi(x) \rightarrow$  state value  $x$   
 $\hat{p} \rightarrow \psi(p) \rightarrow$  state value  $p$

Heisenberg's equation

$$\begin{aligned}\dot{\hat{x}} &= \frac{\partial \hat{H}}{\partial \hat{p}} \\ \dot{\hat{p}} &= -\frac{\partial \hat{H}}{\partial \hat{x}}\end{aligned}$$

Schrödinger equation

$$\hat{H}\psi = E\psi$$

$V(x) + m\ddot{x} = F$  (Newton's equation)  
 $\downarrow$  constant of motion  
 $H = V(x) + \frac{p^2}{2m} \rightarrow \hat{H} = V(\hat{x}) + \frac{\hat{p}^2}{2m}$





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electromagnetic field

$$E = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \omega^2 A^2$$

$$\vec{E} = \frac{\partial \vec{A}}{\partial t} \rightarrow E_i = \dot{A}_i$$

$$\vec{B} = \nabla \times \vec{A} \rightarrow B_i = \epsilon_{ijk} \partial_j A_k$$

$$E = \frac{\epsilon_0}{2} \omega^2 A^2 + \frac{1}{2\mu_0} k^2 A^2$$

$$= \frac{\epsilon_0}{2} \omega^2 A^2 + \frac{1}{2\mu_0} \omega^2 A^2$$

$$\omega = ck \rightarrow E = \frac{\epsilon_0}{2} \omega^2 A^2 + \frac{1}{2\mu_0} \omega^2 A^2 = \epsilon_0 \omega^2 A^2$$

(classical) mechanical oscillator

$$E = \frac{m\omega^2}{2} x^2 + \frac{1}{2m} p^2$$

quantum oscillator

$$\hat{H} = \frac{m\omega^2}{2} \hat{x}^2 + \frac{1}{2m} \hat{p}^2$$

$$= \left( \frac{m\omega}{2} \hat{x} + i \frac{1}{2m} \hat{p} \right) \left( \frac{m\omega}{2} \hat{x} - i \frac{1}{2m} \hat{p} \right)$$

$$\hat{x}^2 + \hat{p}^2 = (x+iy)(x-iy)$$

$$= c c^\dagger$$

$$i(\hat{x}\hat{p} - \hat{p}\hat{x})$$

$$\equiv [\hat{x}, \hat{p}]$$

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$$H = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \epsilon_0 \omega^2 A^2$$

$$\hat{H} = \frac{\epsilon_0}{2} (\hat{E}^2 + c^2 \hat{B}^2)$$

$$= \frac{\epsilon_0}{2} (\hat{E} + ic\hat{B})(\hat{E} - ic\hat{B})$$

Try  $\hat{H}' = \frac{\epsilon_0}{2} (\epsilon_0 \hat{E} + ic\hat{B}\hat{B})(\hat{E} - ic\hat{B}\hat{B})$

Canonical quantization  $E \cdot B = \hat{E} \cdot \hat{B}$   
 $D = \hat{B}\hat{B}$

$$= \frac{\epsilon_0}{2} \left[ (\hat{E}^2 + c^2 \hat{B}^2) + i\epsilon_0 c \hat{B} \cdot (\hat{E}\hat{B} - \hat{B}\hat{E}) \right]$$

This is what we want let  $[\hat{B}, \hat{E}] = -i$ 

$$\rightarrow \hat{H}' = \frac{\epsilon_0}{2} (\hat{E}^2 + c^2 \hat{B}^2) + \frac{\epsilon_0}{2} c \hat{B} \cdot \hat{E}$$

$$\rightarrow \hat{H} = \hat{H}' - \frac{\epsilon_0}{2} c \hat{B} \cdot \hat{E}$$





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$$\textcircled{1} \rightarrow \hat{H} = \frac{\epsilon_0}{2} \underbrace{(\epsilon_0 \hat{E} + i c B_0 \hat{B})}_{\hat{A}} \underbrace{(\epsilon_0 \hat{E} - i c B_0 \hat{B})}_{\hat{A}^\dagger} - \frac{\epsilon_0}{2} c B_0 \epsilon_0 \quad \textcircled{2}$$

$\epsilon_0, \epsilon_0, B_0, c$

Plane wave assumption  
 $\omega = ck$   
 $B_0 = \frac{E}{c}$   
 $E_0 = \omega A_0$   
 $A \rightarrow A_0 \hat{a}$

$$\hat{H} = \frac{\epsilon_0 \epsilon_0^2}{2} (\hat{E} + i \hat{B}) (\hat{E} - i \hat{B}) - \frac{\epsilon_0 \epsilon_0^2}{2}$$

$$= \frac{\epsilon_0 \epsilon_0^2}{2} (\hat{A} \hat{A}^\dagger - 1) = \frac{\epsilon_0 \omega^2 A_0^2}{2} (\hat{A} \hat{A}^\dagger - 1)$$

$$\hat{H} = \frac{\epsilon_0 \omega^2 A_0^2}{2} \hat{A} \hat{A}^\dagger - \frac{\epsilon_0 \omega^2 A_0^2}{2}$$

$$= \frac{\epsilon_0 \omega^2}{2} (\hat{A} \hat{A}^\dagger) - \frac{\epsilon_0 \omega^2 A_0^2}{2}$$

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electromagnetic field operator  $A \rightarrow A_0 \hat{a}$   
 electric field operator  $E \rightarrow \epsilon_0 \hat{E}$   
 magnetic field operator  $B \rightarrow B_0 \hat{B}$

Using  $[\hat{B}, \hat{E}] = -i$

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2} [\hat{E} + i \hat{B}, \hat{E} - i \hat{B}]$$

$$[\hat{E}, \hat{E}] = 0, [\hat{B}, \hat{B}] = 0$$

$$\rightarrow [\hat{a}, \hat{a}^\dagger] = \frac{1}{2} [\hat{E}, i \hat{B}] + \frac{1}{2} [i \hat{B}, \hat{E}]$$

$$= -\frac{i}{2} [\hat{E}, \hat{B}] + \frac{i}{2} [\hat{B}, \hat{E}]$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Therefore,

$$\hat{H} = \epsilon_0 \omega^2 A_0^2 (\hat{a} \hat{a}^\dagger - \frac{1}{2})$$

$$= \epsilon_0 \omega^2 A_0^2 (1 + \hat{a}^\dagger \hat{a} - \frac{1}{2})$$

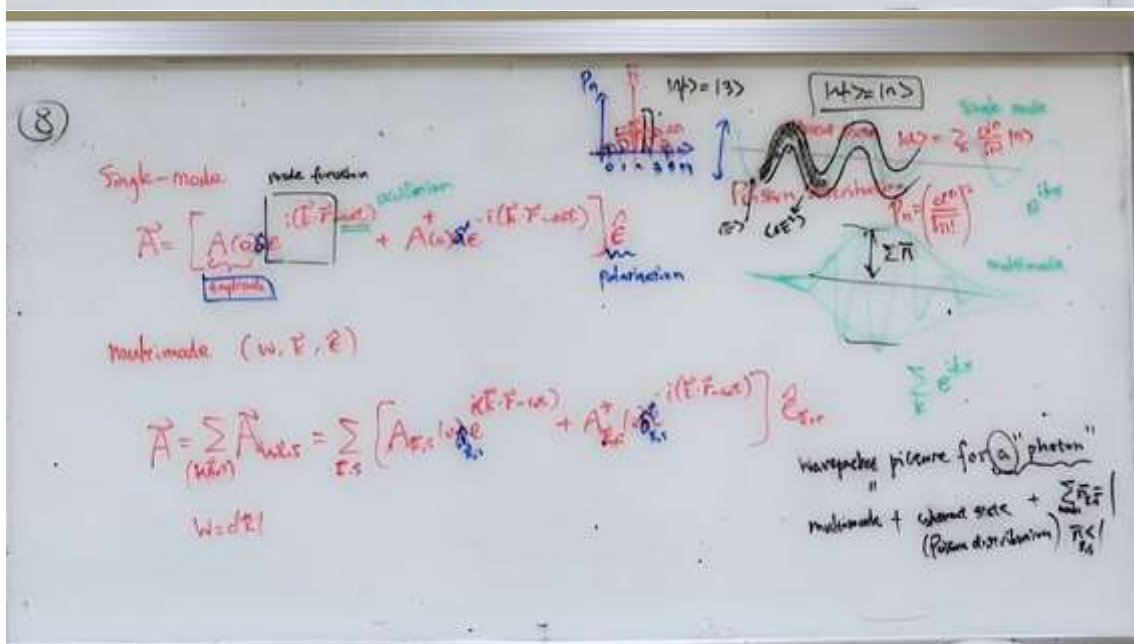
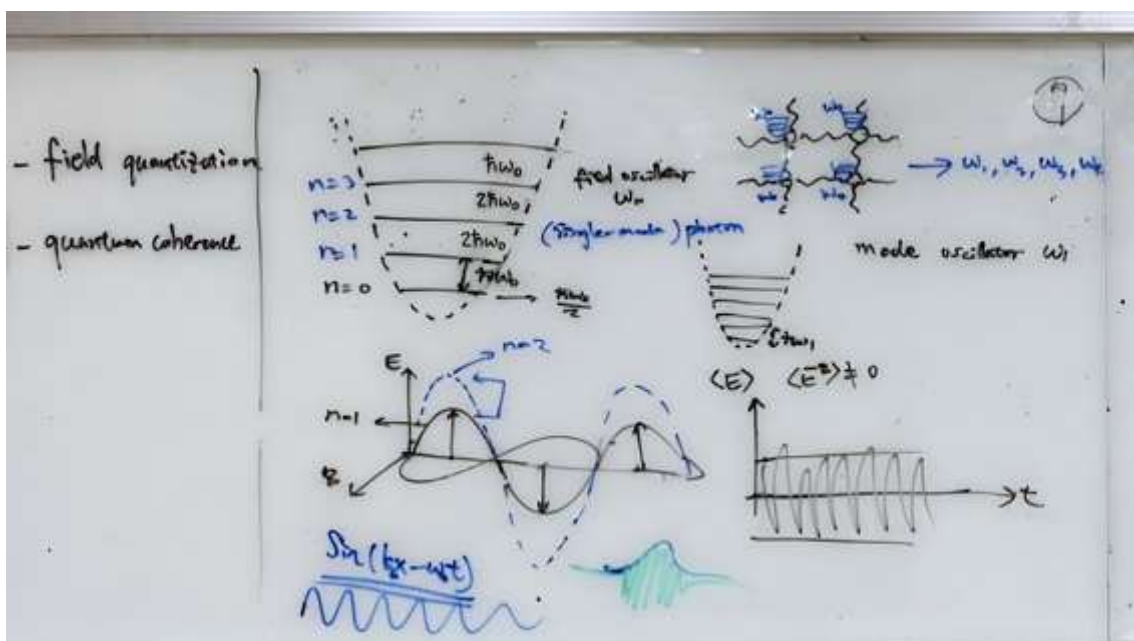
$$= \epsilon_0 \omega^2 A_0^2 (\hat{a}^\dagger \hat{a} + \frac{1}{2}) = \epsilon_0 \omega^2 A_0^2 \hat{a}^\dagger \hat{a} + \frac{\epsilon_0 \omega^2 A_0^2}{2}$$

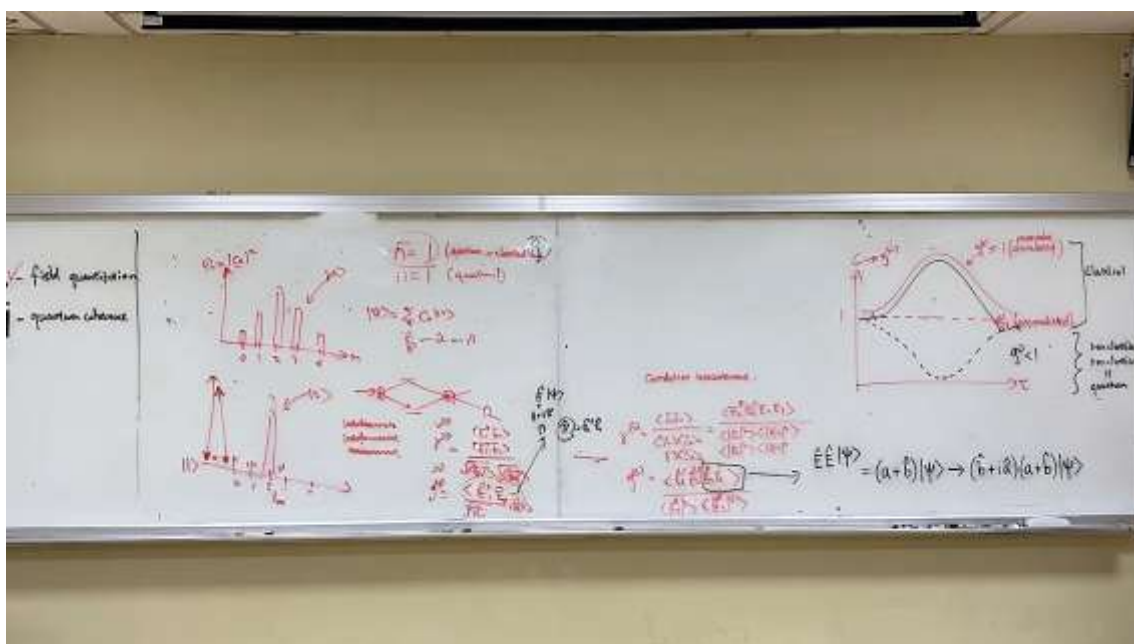
$\hat{H} |\psi\rangle = E_n |\psi\rangle$   
 $E_n = \hbar \omega (n + \frac{1}{2})$   
 $n = 0, 1, 2, 3, 4, \dots$

(vacuum energy) zero fluctuation  $A_0$









- ✓ - field quantization
- quantum coherence

